Final Exam solutions

1. a) $-bv^{3/2} = m\frac{dv}{dt}$ $\int -\frac{dv}{v^{3/2}} = \int \frac{b}{m}\,dt$

$$2v^{-1/2}\Big|_{v_i}^{v_f} = \alpha t \qquad \alpha = b/m$$

$$\frac{2}{\sqrt{v_f}} - \frac{2}{\sqrt{v_i}} = \alpha t \qquad \text{if } v_f = 0 \quad t \to \infty$$

b) $v_f = v(t)$

$$\frac{2}{\sqrt{v_f}} = \alpha t + \frac{2}{\sqrt{v_i}}$$

$$\frac{1}{\sqrt{v_f}} = \frac{\alpha}{2}t + \frac{1}{\sqrt{v_i}} = \frac{\alpha}{2}t + \beta$$

$$\frac{1}{v_f} = \frac{dt}{dx} = \left(\frac{\alpha}{2}t + \beta\right)^2$$

so $$\int dx = \int \frac{dt}{\left(\frac{\alpha}{2}t + \beta\right)^2}$$

$$\Delta x = -\frac{1}{\frac{\alpha}{2}\left(\frac{\alpha}{2}t + \beta\right)}\Bigg|_{t=0}^{t=\infty} \qquad \text{from Eqn sheet}$$

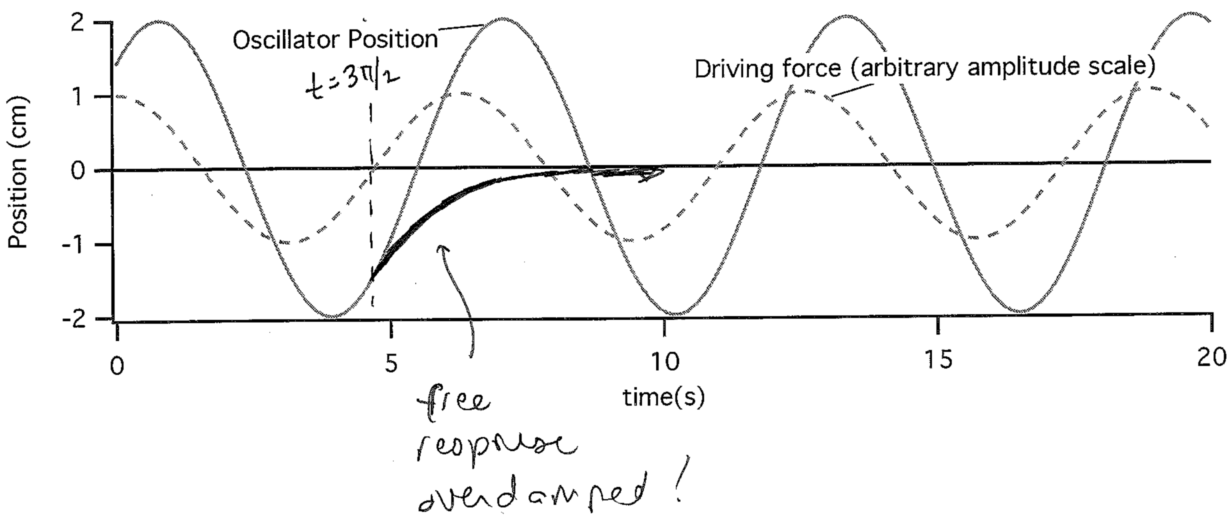
$$= +\frac{2}{\alpha\beta}$$

$$= \frac{2m}{b}\sqrt{v_0}$$

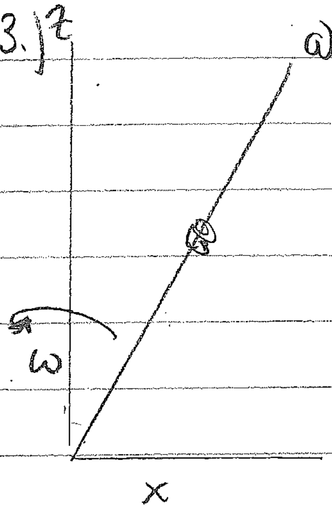
2a) $\delta = \pi/4. < \pi/2. \quad \omega < \omega_0.$

b) $\frac{\pi}{4} = \arctan\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$ so $\frac{2\beta\omega}{\omega_0^2-\omega^2} = 1$

$\frac{\cancel{2}4}{\omega_0^2-1} = 1 \qquad \omega_0^2 = 3 \qquad \omega_0 = \sqrt{3}$



3.



a) $z = \alpha x$

$\mathcal{L} = T - U$

$= \frac{1}{2} m(\dot{x}^2 + \alpha^2 \dot{x}^2) + \frac{1}{2} m x^2 \omega^2 - mg\alpha x$

b) $\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}$

$x\omega^2 - g\alpha = \frac{d}{dt}(\dot{x} + \alpha^2 \dot{x}) = (1+\alpha^2)\ddot{x}$

c) Equ when $x\omega^2 - g\alpha = 0$ $\quad x = \frac{g\alpha}{\omega^2}$.

If $x \uparrow$, $\ddot{x}$ is $+$, so unstable.

d) $(1+\alpha^2)\ddot{x} - \omega^2 x = -g\alpha$

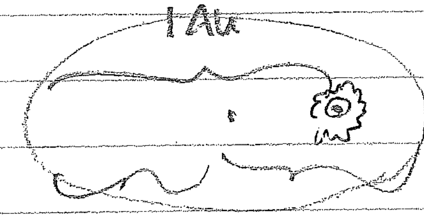
Homogeneous. $x = e^{\pm\beta t}$ $\quad \beta = \frac{\omega}{\sqrt{1+\alpha^2}}$.

Inhomo. $x = +\frac{g\alpha}{\omega^2}$

Complete $x = A_1 e^{\beta t} + A_2 e^{-\beta t} + \frac{g\alpha}{\omega^2}$.

4. a) $\left(\frac{\tau}{\tau_e}\right)^2 = \left(\frac{1}{2}\right)^2 = \left(\frac{a}{a_e}\right)^3$ $a = 0.63\, a_e$.

Closest approach =
$2 \times 0.63 - 1 = 0.26$ AU



b) Aphelion $= \frac{c_{sc}}{1-\epsilon}$ ratio $= \frac{1+\epsilon}{1-\epsilon} = \frac{1\text{ AU}}{0.26\text{ AU}} = 3.85$

Perihelion $= \frac{c_{sc}}{1+\epsilon}$

$1+\epsilon = 3.85 - 3.85\epsilon$

$\epsilon = 2.85/4.85 = 0.59$

c) $\frac{c}{1} = \frac{c_{sc}}{1-0.59}$ $c_{sc} = 0.41c$

d) $\frac{c_{sc}}{c} = \frac{\ell^2}{\ell_0^2}$ $\ell = \sqrt{.41}\,\ell_0 = 0.64\,\ell_0$

e) Since $\vec{\ell} = \vec{r} \times \vec{p}$ and r is constant during thrust,
v must be $= 0.64\, v_e$

f) $\Delta V = 0.36\, V_e = 10.7$ km/s
$V_{ex} = 3$ km/s.

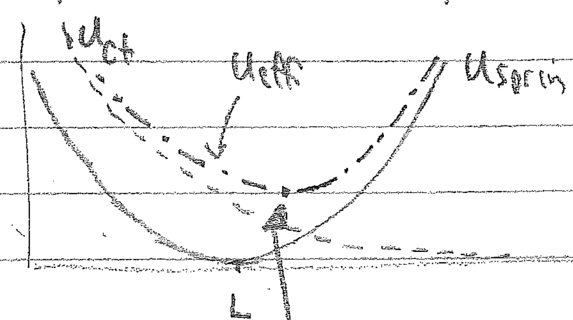
Rocket eqn: $\Delta V = V_{ex} \ln\left(\frac{M_0}{M}\right)$ $\ln \frac{M_0}{M} = \frac{10.7}{3} = 3.5\overline{8}$

$\frac{M_0}{M} = 35.8$ 97% fuel.

5. a) $\vec{p} = m\vec{v}$

b) About C.M. $\vec{\ell} = \vec{r} \times \vec{p}$ $|\ell| = \frac{L}{2} \cdot mv$.

c) $U_{eff} = U + U_{cf} = U + \dfrac{\ell^2}{2\mu r^2}$ $U = \frac{1}{2}k(r-L)^2$



d.) Yes. Here

Easiest to set $F = 0 = -k(r-L) + \dfrac{\ell^2}{\mu r^3}$

if $r = (L+\delta)$ we find

$$0 = -k\delta + \frac{\ell^2}{\mu}\frac{1}{L^3}\left(1+\frac{\delta}{L}\right)^{-3} \approx -k\delta + \frac{\ell^2}{\mu L^3}\left(1 - \frac{3\delta}{L}\right)$$

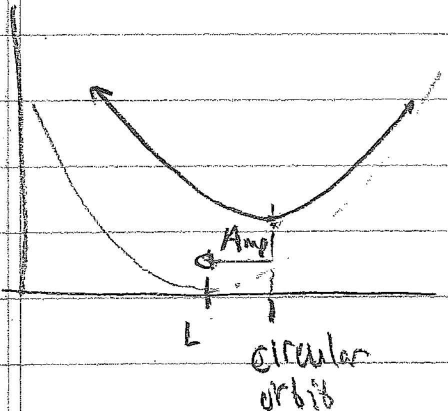
$$\left(k\mu L^3 + \frac{3\ell^2}{L}\right)\delta = \ell^2$$

$$\boxed{\delta = \frac{\ell^2 L}{k\mu L^4 + 3\ell^2}}$$ Recall $\ell = \dfrac{Lmv}{2} = L\mu v$.

Can ignore this in small ℓ limit.

$r_{stp} = L + \delta$.

5e) the initial kick has $\dot{r}_i = 0$ (it's $\perp$ to r.)
So the amplitude is just the initial amplitude



$A = \delta.$

f) Both U_{spring} and U_{cf} are concave up. their sum will thus be even steeper upward, and $\omega \uparrow$. $\omega > \sqrt{\frac{k}{\mu}}$.

g) $k_{eff} = -\frac{dF}{d\delta}$ (this is also the 2nd derivative of U_{eff})

$$= -\left(-k - \frac{3\ell^2}{\mu L^4}\right) = k + \frac{3\ell^2}{\mu L^4} \qquad \omega = \sqrt{\frac{k_{eff}}{\mu}}$$

h) $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{M}{2} = 1\,kg$ $\quad v = 1\,m/s.$ $\quad \ell = L\mu v = 1 \cdot 1 \cdot 1 = 1.\ kg\frac{m^2}{s}$

So $k_{eff} = 13 + \frac{3 \cdot 1}{1 \cdot 1} = 16 \qquad \omega = \sqrt{\frac{16}{1}} = 4\ s^{-1}$

The whole system rotates with angular frequency $\frac{v}{L} = 1\,s^{-1}$.

Orbit: